**Problem Sets for math 436**

**Unit 1**

1. Multiply 29x15. Does this resemble anything familiar?
2. Use the method of false position to solve:
3. 2x + 5 x = 44
4. n + 3 n + 6 = 50

Note that the order of operations becomes important here.

1. The Egyptian formula for the area of a circle with diameter d is A = ((8/9)(d))2. This would say that the Egyptian equivalent of π is 3.16049
2. Give some possible explanations.
3. Using 64 pennies show that a circle of diameter 9 can be filled with these pennies whose diameter is 1 each.
4. Show that the same 64 pennies can fill an 8 x 8 square.
5. Conclude that the A of the circle is as described by the Egyptians.
6. Conjecture why 60 was used as the base of the number system.
7. Write the number 236 in base 60 using the symbols given.
8. Multiply the number 23 by 16.
9. Solve by the method of false position: A quantity and its 1/7 when added together become 19. What is the quantity?
10. Calculate a quantity such that if it is taken two times along with the quantity itself, the sum comes to 9.
11. The work of a man in logs: the amount of his work is 100 logs of 5 handbreadths diameter, but he has brought them in logs of 4 handbreadths diameter. How many logs of 4 handbreadths in diameter are there?
12. Convert the numbers to sexagesimal notation (Babylonian): 7/5, 13/15, 11/24, and 33/50.
13. Show that the area of the Babylonian “bull’s eye” is given by A = (9/32)a2 where a is the length of the arc (one-third of the circumference). Also show that the length of the long transversal of the barge is (7/8)a and the length of the short transversal is (1/2)a. (Use the Babylonian values of A = C2/12 for the area of the circle, and 7/4 for √3. The Babylonian bull’s eye is made up of 2 arcs each 1/3 of the circumference.
14. Show that the area of the Babylonian barge is given by A = (2/9)a2 where a is the length of the arc that is ¼ of the circumference of the circle. Also show that the length of the long transversal of the barge is (17/18)a and the length of the short transversal is (7/18)a. Use A = C2/12 and √2 = 17/12. The Babylonian barge is made up of 2 arcs each ¼ of the circumference.
15. Solve the problem from the Old Babylonian tablet: The sum of the areas of two squares is 1525. The side of the second square is 2/3 that of the first plus 5. Find the sides of each square.
16. One of two fields yields 1/3 sila per sar, te second yields1/2 sila per sar. The sum of the yields of the two fields is 1100 sila., the difference of the areas of the two fields is 600 sar. How large is each field? Sar measures area and sila measures capacity,
17. Given a circle of circumference 60 and a chord of length 12, what is the perpendicular distance from the chord to the circumference? Use π = 3.
18. You have a rectangle divided into smaller rectangles. You know that each smaller rectangle has integral length or integral width. Prove that the original rectangle also has integral length or integral width. (Hard!)
19. I need to enter a castle that is surrounded by a five meter wide rectangular moat filled with water. I have an aversion for getting wet and only have two 4.8 meter long planks with no rope, nails, etc. How do I get across?
20. Give a geometric justification for the solution of x2 + bx = c.
21. Give a geometric argument to justifies the Babylonian quadratic formula that uses the “quadratic formula” solution to x2 – ax = b.

What happens if the coefficient of the square term in problems 7 and 8 is 3?

**\_\_\_\_\_\_\_\_\_\_\_Unit 2\_\_\_\_\_\_\_\_\_\_\_\_**

1. Thales is said to have invented a method of finding distances of ships from shore by use of the angle-side-angle theorem. Suppose A is a point on the shore and S is a ship in the ocean. Show how you would find the distance from A to S.
2. Thales also is said to have been on top of a tower on the shore with an instrument made of a straight stick and cross piece AC that could be rotated to any desired angle. One rotates AC until one sights the ship S, then turns and sights an object T on the shore without moving the crosspiece. Show how this would measure the distance from A to S.
3. Show how to calculate the distance between two inaccessible points A, B by the use of similar triangles. (Assume for example that you are on one side of a river and the two points are on the other.)
4. Show that any square number is the sum of two consecutive triangular numbers.
5. Find relationships between square numbers, triangular numbers, and oblong numbers.
6. What are the pentagonal numbers? Can you give a general formula for the nth pentagonal number? What is it’s gnomon?
7. Show using dots that 8 times any triangular number plus 1 makes a square number. Show this result algebraically too. (5)
8. Show using dots that any odd square number diminished by 1 makes 8 times a triangular number. Show this result algebraically too.(5)
9. Construct 5 pythagorean triples using the formula for n odd and 5 using the formula for n even.(6)
10. How can seven trees be planted so that there are six rows of trees in a straight line with each row having three trees?
11. Show how to compute the square root of (ab) for any a and any b if you have a stick of length 1 unit.
12. Consider a pentagram. There are five disjoint triangles initially. Just count the five ‘caps’ above the pentagon, do not count the larger triangles formed by using three of the five vertices, for those ‘triangles’ have lines going through them. By adding just two lines, you can go from five to 10 disjoint triangles. How?
13. If you have a circle and you are only given 3 lines to divide it in 7 pieces, how can you accomplish this task?

**\_\_\_\_\_\_\_\_\_\_\_\_Unit 3\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

1. Given that you can construct an equilateral triangle, and a regular pentagon, how do you construct a regular 15-gon. Remember you have only a straight edge and a compass.
2. If a weight of 8 kg is placed 10 m from the fulcrum of a lever and a weight of 12 will the lever incline?
3. Use calculus to prove that the area of a parabolic segment is 4/3 the area of the inscribed triangle. (8)
4. Determine the equations of the parabola and hyperbola whose intersection provides the solution to the cubic equation 3ax2 - x3 = 4a2b. Sketch the two curves on the same pair of axes. (12)
5. Show that for the curve y2 = px, the value of p represents the length of the latus rectum, the straight line through the focus perpendicular to the axis. (13)
6. Show analytically that the vertex of a parabolic segment is that point on the curve whose perpendicular distance is the greatest.
7. Write out the expansion of π using: “May I have a large container of coffee”.
8. How many correct decimal places of π does the following mnemonic yield – write it out:

Sir, I bear a rhyme excelling

Celestial sprites elucidate

All my own striving can’t relate.

**\_\_\_\_\_\_\_\_\_\_\_Unit 4\_\_\_\_\_\_\_\_\_\_\_\_**

1. There is a story about Archimedes that he used a “burning mirror” in the shape of a paraboloid of revolution to set fire to enemy ships in the harbor. What would be the equation of the parabola that one would rotate to form the appropriate paraboloid if it were to be designed to set fire to a ship 100 m from the mirror? How large would the mirror have to be? What is the likelihood that the story is true? See notes.
2. Prove (a+b)2 = a2 + 2ab + b2  geometrically.
3. Prove (a – b)2 = a2 - 2ab + b2 geometrically.
4. Show when a + b/2a is an approximate square root of a2 + b.
5. Write the equation given for an ellipse in standard form for an ellipse by completing the square.
6. Write the standard form for a hyperbola from the equation given for the hyperbola.
7. Use calculus to give a proof that the line from the focus to a point on a parabola makes an angle with the tangent at that point equal to the angle made by a line parallel to the axis.
8. Show analytically that the solution to the three-line locus problem is a conic section in the case where two of the lines are parallel and the third is perpendicular to the other two. Characterize the curve in reference to the distance between the two parallel lines and the given ratio.
9. Show analytically that the solution to the general three-line locus problem is always a conic section.
10. Calculate crd(30), crd(15), crd(7.5) using the Hipparchus half angle formula starting with crd(60) = R = 60.
11. Use Ptolemy’s difference formula to calculate crd(6) and crd(3).
12. Calculate the rising times for ϕ = 45 and λ = 60 and 90.
13. Calculate the length of daylight on a day when λ = 60 at latitude 36. Calculate the local time of sunrise and sunset.
14. At what date does the midnight sun begin at latitude 75?

**\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_Unit 5\_\_\_\_\_\_\_\_\_\_\_**

1. Determine Diophantus’ age at his death from this epigram:

*“The tomb holds Diophantus and tells scientifically the measure of his life. God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, He clothed his cheeks with down. He lit him the light of wedlock after a seventh part, and 5 years after his marriage He granted him a son. Alas, late-born wretched child, after attaining the measure of half his father’s life, chill fate took him. After consoling his grief by this science of numbers for four years, He ended his life”.*

1. Solve Diophantus’ problem II-10: To find two square numbers having a given difference. Diophantus puts the given difference at 60. Also**, give a general rule** for solving this problem with any given difference b.
2. Solve Diophantus’s problem B-9: To divide a given number into two parts such that the sum of their cubes is a given multiple of the square of their differences. i.e x + y =a and x3 + y3 = b(x – y)2. Diophantus took a = 20 and b = 140.
3. Solve Diophantus’ problem IV-9. To add the same number to a cube and its side and make the second sum the cube of the first. (The equation is x + y = (x3 + y)3; he started by letting x = 2z and y = 27z3 – 2z.
4. Write the equation of the locus of points for the four line problem where three of the lines are parallel to each other and the fourth is perpendicular to the 3. The locus of points such that d1d2 = kd3d4 for some k. (Hint: It does not make any difference how you label the four lines.)
5. Provide an analysis of: If a straight line is cut in extreme and mean ratio, the sum of the squares on the whole and on the lesser segment is triple the square on the greater segment.
6. Solve the epigram 130: Of the four spouts, one filled the whole tank in a day, the second in two days, the third in three days, and the fourth in four days. What time will all 4 spouts take to fill it?
7. Compare Diophantus’s use of the method of false position to that used by the Egyptians and the Babylonians.
8. Solve the Epigram: A: Give me ten coins and I have 3 times as many as you. B: And if I get the same from you, I have five times as much as you. How many coins does each have?
9. You have 10 cans of peas. The cans are open. In each can, there are 100 peas. In nine of the 10 cans, each pea weighs one gram. In the tenth can, each pea weighs only 0.9 grams. You do not know which can has the smaller peas, nor is it possible to tell with the naked eye. To help you, you have an electronic scale. However, the scale is not in great shape, and can only provide one correct measurement before permanently malfunctioning. How can one, using only the one measurement afforded by the scale, determine beyond a shadow of a doubt which can it is that has the smaller peas?

**\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_Unit 6\_\_\_\_\_\_\_\_\_\_\_**

1. Try the Chinese method to compute the square root of 78900
2. Use the Chinese square root algorithm to find the square root of 12,812,904.
3. There is a reservoir with five channels bringing in water. If only the first channel is open, the reservoir can be filled in 1/3 of a day. The second channel by itself can fill the reservoir in 1 day, the third channel in 2.5 days, the fourth one in 3 days and the fifth one in 5 days. If all of the channels are open together how long will it take to fill the reservoir?
4. A pole of unknown length leans against the wall so that its top is even with the top of the wall. If the bottom of the pole is moved 1 foot further from the wall, the pole will fall to the ground. How long is the pole?
5. A deep well 5 ft in diameter is of unknown depth (to the water level). If a 5 foot post is erected at the edge of the well, the line of sight from the top of the post to the edge of the water surface below will pass through a point .4 ft from the lip of the well below the post. What is the depth of the well?
6. Use the Chinese Remainder Theorem to solve: If N=0(mod3) and N= 1(mod 4), what is N?
7. Use the Chinese Remainder Theorem to solve: Find N if N= 0(mod 11), N = 0(mod5), N= 4(mod9), N= 6(mod8), and N= 0(mod7).
8. Consider an army with 10 generals. One wants a security system such that any three of them can determine the code to launch nuclear missiles, but no two of them can. It is possible to devise such a system by using a quadratic polynomial, such as a x^2 + bx + c; to launch the missiles, one must input (a,b,c). One cannot just tell each general one of a, b, or c (as then it is possible that some subset of three generals won’t know a, b and c); however, if a general knows two of (a,b,c), then a set of two generals can launch the missiles! What information should be given to the generals so that any three can find (a,b,c) but no two can?
9. You have 25 horses and want to know which are the three fastest. Whenever you race horses, the order of finish accurately reflects the relative speeds of the horses but you can only race five at a time. What’s the minimum number of races required to determine the three fastest, and how do you do it?

**\_\_\_\_\_\_\_\_\_\_\_Unit 7 \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

1. Solve the congruence N= 10 mod 137 by both the Chinese method and the Brahmagupta’s procedure and compare the two methods.
2. Solve the indeterminate equation 17n-1 = 75m by both the Chinese and Indian procedures using the Euclidean Algorithm explicitly.
3. We have the following problem from Mahavtra. There are 4 pipes leading into a well. Among these each fills the well (in order) in ½, 1/3, ¼, and 1/5 of a day. In how much of a day will all of them together fill the well and each of them to what extent?
4. Two travelers found a purse containing money. The first said to the second. “By securing half of the money in this purse, I shall become twice as rich as you.” The second said to the first. “By securing two-thirds of the money in the purse, I shall with the money I have on hand, have three times as much money as what you have on hand.” How much did each have and how much was in the purse?
5. Solve the problem N = 5mod6 = 4mod5 = 3mod4 = 2mod3 by the Indian procedure.
6. Compare the Indian method for this problem to the Chinese procedure used on this problem.
7. Solve the problem N = 5mod6 = 4mod5 = 3mod4 = 2mod3 by the Indian procedure. Compare this method for this problem to the Chinese procedure used on this problem.

**\_\_\_\_\_\_\_\_\_\_\_\_\_Unit 8\_\_\_\_\_\_\_\_\_\_\_\_\_**

1. Divide x3 + 5x2 – 3x + 6 by x + 2.
2. Solve (x + √½ x)2 = 4x
3. Show that the solution to x3 + 200x = 20x2 + 2000 is achieved by finding the intersection of a hyperbola and a semicircle. Generalize it to find the solution of x3 + cx = bx2 + d.
4. Show directly without using Ptolemy’s theorem that in an isosceles trapezoid, the square on a diagonal is equal to the sum of the product of the two parallel sides plus the square on one of the other sides.
5. Discuss various ways of classifying roots of a cubic. Can you come up with four?
6. Given the quadrilateral inscribed in a circle with ab – ag – 10 and bg = 12, find the diameter ad of the circle:

 a

 b g

 d

1. A merchant sells four drugs. The cost of the first drug is 2 dinars per litra; the cost of the second is 3 dinars per litra; the cost of the third drug is 12 dinars per litra; the cost of the fourth drug is 20 dinars per litra. How many of each drug should one buy so the cost for each is the same?
2. Prove that the difference of the squares of two consecutive triangular numbers is a cube. Hint: factor the difference of squares as a sum times a difference.
3. If 7 rolls of pepper are worth 4 bezants and 9 pounds of saffron are with 11 bezants, how much saffron will he have for 23 rolls of peppers?

**\_\_\_\_\_\_\_\_\_\_\_Unit 9\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

1. Divide 10 into 2 parts so that the product is 13 + √128

1. Solve 6x3 = 43x2 + 79x + 30
2. Solve x3 – 3x = 10 (Cardano)
3. Use Ferrari’s method to solve x4 + 4x + 8 = 10x2
4. Solve for x and y: xy = 8 and x2 + y2 = 27
5. Make a perspective drawing of a checkerboard. First establish a reasonable distance for the vanishing line and vanishing point and then construct the horizontal lines using the rules in the text.
6. Put telephone poles in the picture given. If the height at the outset is 15ft what is the height further down? Assume the distance from ground to the vanishing point is 100 ft.
7. Solve the problem from On Triangles of finding two sides AB, AG of a triangle ABG given that BG = 20, the perpendicular AD = 5, and the ratio AB:AG = 3:5. See figure in the book on p. 253. E is the point such that DE = EBD and set EG = 2x.
8. In triangle ABC suppose the ratio Angle A:angle B = 10:7 and

the ratio angle B:angle C = 7:3. Find the three angles and the ratio of the sides.

1. Each duke has twice as many earls as there are dukes. Each earl has twice as many soldiers as there are dukes. 1/200 the sum of all men is 9 times the number of dukes.

**\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_Unit 10\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

1. xy = c is a hyperbola. Show that xy + c = rx + sy is also a hyperbola. Give the asymptotes.
2. Pro ve C(n,k) : C(n,k+1) = (k+1) : (n-k) in 2 ways
3. Find the locus of points for the equation b2 – 2x2 = 2xy + y2
4. Solve x3 = 300 x + 432 given that x = 18 is one solution using Girard’s technique.
5. Prove by induction on n that: C(n,k) = ∑C(j,k-1) summed from j=k-1 to n-1 and k<n.
6. Show that the odds of throwing a 6 in 4 throws of a die is 671/625.
7. On a toss of 2 dice, I win with 7 total, you win with a 6 total, What is the ratio of chances that **I win over you** if **you start first**?
8. There is a game with two players and they throw a pair of dice. Player 1 wins if the sum is 7 and player 2 wins if the sum is 6. The stakes are split if neither player wins. What is the expectation of each player? Suppose the game continues until someone wins. What is the probability that each will win if player 2 starts?
9. We have 12 balls, four of which are white and eight are black. Three blindfolded players, A, B, C, draw a ball in turn, first A, than B, then C. The winner is the one who first draws a white ball. Assuming that each (black) ball is replaced after being drawn, fin the ratio of the chances of the three players. Show your work.
10. 100 people are waiting to board a plane. The first person’s ticket says Seat 1; the second person in line has a ticket that says Seat 2, and so on until the 100th person, whose ticket says Seat 100. The first person ignores the fact that his ticket says Seat 1, and randomly chooses one of the hundred seats (note: he might randomly choose to sit in Seat 1). From this point on, the next 98 people will always sit in their assigned seats if possible; if their seat is taken, they will randomly choose one of the remaining seats (after the first person, the second person takes a seat; after the second person, the third person takes a seat, and so on). What is the probability the 100th person sits in Seat 100?
11. Three players enter a room and a red or blue hat is placed on each person’s head. The color of each hat is determined by a coin toss, with the outcome of one coin toss having no effect on the others. Each person can see the other players’ hats but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, the players must simultaneously guess the color of their own hats or pass. The group shares a hypothetical $3 million prize if at least one player guesses correctly and no players guess incorrectly. Find a strategy for the group that maximizes the chances that the group will win the $3million dollars.
12. There is a game with two players and they throw a pair of dice. Player 1 wins if the sum is 7 and player 2 wins if the sum is 6. The stakes are split if neither player wins. What is the expectation of each player? Suppose the game continues until someone wins. What is the probability that each will win if player 2 starts?

**\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_Unit 11\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

1. Use one of Fermat's methods to find the maximum of bx - x3. How would Fermat decide which of the two solutions to choose as his maximum?
2. . Use Descartes's circle method to determine the slope of the tangent line to y2 = x.
3. Find the slope of y = xn at (x0, x0n) using Hudde’s rule applied to Descartes’ method.
4. Use Barrow's a; e method to determine the slope of the tangent line to the curve x3+y3 = c3.
5. Write 1/(1+x) as a power series using the Newton method i.e divide 1 by 1+x using long division.
6. Using the algorithm to determine square roots, find a power series for √(1+x2).
7. Calculate a power series for √(1+x).
8. Calculate a power series for 1/(1-x2)
9. Compare and contrast the calculuses of Newton and Leibniz in terms of their notation, their ease of use, and their foundations.